Teaching and learning elementary mathematics

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The current dilemma: three false dichotomies:

**Managerialist:** Global “competencies” *versus* “content”  
(Mathematics is the most important “21st century competency”)

**Democratic:** “Progressive left” *versus* “reactionary right”

**Mathematics education:** Child-centred *versus* subject-centred  
“bottom-up” *versus* “top-down”

**Experience:** “Think. Naively taking sides leads to trouble!”

Learning and teaching mathematics more effectively is hard. But – given thought and cooperation, it is possible.

Portuguese mathematicians and educators have to choose:  
• To work (together) to develop the (mathematics) curriculum?  
• Or to trust some external “21st century” Snake Oil – such as “definitions of concepts adopted by OECD”?  

**Currículo Século XXI**

Conferência Internacional  
Pensar a Matemática

**REPUBLICA**

**PORTUGUESA**

**EDUCAÇÃO**
TIMSS and PISA: Learn. But don’t worship PISA scores. And don’t just accept PISA’s claims/OECD “competencies”.

Portugal has seen dramatic change/improvement since 1995. The recent TIMSS/PISA results confirm this - though parties should hesitate to claim direct “responsibility” (England 2003-15)

Do not be seduced by slogans, or by reforms elsewhere (e.g. BC, Canada). The modern world is very different from the one I grew up in; but bringing up children, learning language and reading, internalising place value, calculating with numbers, with symbols, and with geometric shapes, etc. require one to go through more-or-less the same stages as before. In short:
- To access “21st century skills” one has to start with familiar 19th/20th century pathways.
There is no “21st century short cut”.
Brief educational CV

1968: New math for teachers in Tanzania
1975-82: Saturday maths school to reshape the curriculum.
1978-80: Visits to Freudenthal & IOWO (now often misrepresented).
1979-2000: School Mathematics Project – the only academic in a writing team for secondary textbooks (used in 60% of schools).
1980s: Start local and national secondary school “challenges”.

My original goal: “Learn from other countries to find effective alternatives to traditional instruction.”

What became increasingly clear:
- “Top-down” and “bottom-up” viewpoints must cooperate.
- Naive differentiation (acceleration for some; well-meaning bottom-up schemes for others) undermines progression for all.
- Everyone can and must grasp the basic abstractions.
An example very like Portugal(P): namely UK/England(E).

- TIMSS Grade 4: P +100; E +50 points since 1995
- TIMSS Grade 8: P improves slightly; E improves not at all

England Huge effort at primary level (2000) – and real “change”. But purely “bottom-up”, so “backward-looking”. We failed to identify and lay the foundations for secondary mathematics. (Wales chose a different, gentler route: “interdisciplinary”, “projects”, etc. Result: very, very bad news for Wales!)

A balanced curriculum has to focus on achieving serious mathematical goals at age 16/18 for large numbers. So it has to respect the structure of mathematics, while also respecting the way young minds internalise abstract mathematics.
At primary level in England, the most striking example of this combined “top-down + bottom-up” approach is to be found in
- the links with Shanghai, and
- the primary *Singapore textbooks project*, which is having a dramatic impact on both pupils *and teachers.* Both are helping us to rediscover *didactics* (how to prepare the ground; how to introduce ideas/methods; how to use a few consistent models to support cumulative internalisation).

The approaches
- are “open” (but focused to establish the “target method”)
- are “universal” (whole classes, very little differentiation)
- emphasise simple structures, in the maths and the learning
- focus on “variations” (what is needed for progression).

Neither approach has been extended beyond Grade 5.
Key ingredients of any successful approach (?)

- Classes are best taught as a (social/communal) whole.
- Key ideas are accessible to all if taught in a structured way.
- Mathematics is inescapably abstract, but does not start thus: hence the C-P-A approach, with C-P as stepping stone to A

- One cannot separate “process” from content: there are no “competencies”, or “21st century skills” that can be taught separately from mathematics. Hence the first task is
  - to organise the mathematical/didactical sequence
  - then experiment to find effective ways to teach it.

- Find a didactical structure that works, and develop it (most additions, such as “pedagogical differentiation, experimental activities, projects, and interdisciplinary work” make it hard for pupils/teachers to see where they are going.
Content and style

There is widespread agreement about much early content; yet it is surprisingly hard to draft a coherent curriculum. This is partly due to the superficial understanding of many teachers, educators and mathematicians as to what aspects of each topic are essential for subsequent progression.

This superficiality becomes more marked at secondary level.

Nevertheless, one can draft a potential curriculum as a basis for discussion (as indicated by Part III of my book: Teaching mathematics at secondary level, Open Book, 2014).

“Top-down” mathematicians may try to hurry certain stages; and “bottom-up” educators may need to reconsider the importance of themes they are inclined to dismiss. But they must work together to reach a consensus.
What cannot work is
(i) to add more content than can be effectively introduced, understood, and mastered in the available time, or
(ii) to take a carefully planned (but perhaps too demanding) sequence and simply “cut 25%” to make more time available.

The mathematics curriculum is not an expanding suitcase into which more and more material can be stuffed.
Nor is it a piece of string, which can be shortened by cutting.

It is a structure that can be extended or slimmed down - but must then be redesigned. Hence one can try to reduce the curriculum – but be prepared to discover that some of what you try to remove is essential for parts that you wish to retain. Some strands may have been included early for bad reasons (e.g. Data), may now be seen to be premature, so can be revised/delayed.
Language: Early years mathematics, and its associated logic, have their roots in language: the “C” in C-A-P often involves focused group discussion of an example or problem situation. And at every stage, the link between mathematics and language needs to be strengthened through Word Problems.

Number and measures: All pupils need to master the “place value” system. This requires that they become friends
• with the numbers 1-5, then 1-10 (and “0”)
• with the associated number-words, and
• with combinations/differences; addition/subtraction later using these as digits in numeral form (29; 4562; 7,093).

The next sequence
  - multiplication/division (including written algorithms), fractions, decimals, negative numbers, powers/roots/surds, irrationals, ...

needs to be carefully sequenced – linking with measures from the outset, and preparing for algebra.
**Structural arithmetic:** Calculation needs to exploit place value and to become robust and varied. Upper primary must move beyond “blind calculation with larger and larger numbers” to focus on structure – exploiting place value (73 + 39 + 27 = ?, 12 × 75 = ?, ...) and – algebraic structure (7×29 + 3×29, 3^2 × 4^2, 0.144 ÷ 1.2, ...) in preparation for algebra.

**Elementary number theory** Work with fractions and structure draws attention to squares, cubes; factors, common factors; primes, prime factorisation; etc. (to be handled in the spirit of a sensitive teacher rather than a pure mathematician!)

**Measures and geometry:** Measures are central to number, but also feature in geometry, which has a dual nature – part numerical, and part purely geometrical. How this is treated requires a consensus about the secondary mathematics to which it leads (fractions, algebra, functions and graphs, euclidean geometry, etc.).